Convergence Tests of Infinite Series

Vikhyat Korrapati

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This document can be viewed in HTML format at http://vikhyat.net/articles/convergence_tests/.

1 Positive Term Series

1.1 Integral Test

Let $\sum a_n$ be a positive term series, and let $a_n = f(n)$ such that $f(n)$ decreases as n increases. Then $\sum a_n$ converges or diverges if $\int_1^\infty f(x)dx$ is finite or infinite respectively.

1.2 p-Series Test

Let $\sum a_n$ be a positive term series given by $a_n = \frac{1}{n^p}$. Then, $\sum a_n$ is convergent if $p > 1$, and divergent if $p \leq 1$.

1.3 Comparison Test

Let $\sum a_n$ be a positive term series, then:

1. $\sum a_n$ is convergent if $\sum b_n$ is another convergent series with $a_n \leq b_n$.
2. $\sum a_n$ is divergent if $\sum d_n$ is another divergent series with $a_n \geq d_n$.

1.4 Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two positive term series.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite and non-zero positive quantity, then $\sum a_n$ and $\sum b_n$ will converge and diverge together.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ is divergent, then $\sum a_n$ is also divergent.

1.5 D'Alembert's Ratio Test / Ratio Test

Let $\sum a_n$ be a positive term series, and let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$.

1. The series is convergent if $r < 1$.
2. The series is divergent if $r > 1$ or if r is infinite.
3. The test fails if $r = 1$.

1.6 Cauchy's Root Test / Root Test

Let $\sum a_n$ be a positive term series, and $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = r$.

1. The series is convergent if $r < 1$.
2. The series is divergent if $r > 1$.
3. The test fails if $r = 1$.

1.7 Raabe's Test

Let $\sum a_n$ be a positive term series, and $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = k$.

1. The series is convergent if $k > 1$.
2. The series is divergent if $k < 1$.
3. The test fails if $k = 1$.

1.8 Logarithmic Test

Let $\sum a_n$ be a positive term series, and $\lim_{n\to\infty} n \log\left(\frac{a_n}{a_{n+1}}\right) = k$.

1. The series is convergent if $k > 1$.
2. The series is divergent if $k < 1$.
3. The test fails if $k = 1$.

2 Alternating Series

2.1 Leibniz's Test

If the series $\sum(-1)^n a_n$ is an alternating series, then the series is convergent if:

1. Each term is numerically lesser than the preceeding term. ($|a_{n+1}| < |a_n|$)
2. $\lim_{n\to\infty} a_n$ must equal 0.